Randomized Rounding for Congestion Minimization[1]

- In this lecture, we show how the *Chernoff Bound*, one of the must-know-facts about random variables, helps in analyzing randomized rounding algorithms. Before we begin, let us state the Chernoff bound; we don't prove it in these notes.

Theorem 1 (Chernoff Bounds)**.** Let $X_1, X_2, \ldots, X_n$ be ***independent*** *Bernoulli* random variables with each $X_i \in \{0, 1\}$. Let $X = \sum_{i=1}^{n} X_i$. Then, for any $\varepsilon \in (0, 1)$,

$$\mathbf{Pr}[X \leq (1-\varepsilon)\,\mathbf{Exp}[X]] \leq e^{-\frac{\varepsilon^2\,\mathbf{Exp}[X]}{2}} \tag{LT}$$

and

$$\mathbf{Pr}[X \geq (1+\varepsilon)\,\mathbf{Exp}[X]] \leq e^{-\frac{\varepsilon^2\,\mathbf{Exp}[X]}{3}} \tag{UT1}$$

For the "upper tail", that is for "larger" deviations, we have when $1 \leq t \leq 4$, we have the following (changing ε to t so as to underscore that the deviation is big)

$$\mathbf{Pr}[X \geq (1+t)\,\mathbf{Exp}[X]] \leq e^{-\frac{t^2\,\mathbf{Exp}[X]}{4}} \tag{UT2}$$

and for $t > 4$ (really large), we have

$$\mathbf{Pr}[X \geq (1+t)\,\mathbf{Exp}[X]] \leq e^{-\frac{t \ln t\,\mathbf{Exp}[X]}{2}} \tag{UT3}$$

Remark: *Equations (UT1) to (UT3) hold with* all $\mathbf{Exp}[X]$ *occurrences replaced by* any *upper bound* $\mathbf{Exp}[X] \leq U$. *Equation (LT) holds when* $\mathbf{Exp}[X]$ *is replaced by* any *lower bound* $\mathbf{Exp}[X] \geq L$.

- *Congestion Minimization.* Consider you are designing an integrated chip, that teeny-weeny thingy of which there are hundreds of inside the device you are currently reading these notes on. In such a chip, one has to often connect *pins* by *wires*, and because of design/engineering constraints, between any two pins there are only a finite number of lay-outs in which these wires can be laid down. The chip designer has to decide on one of these lay-outs per pair. However, it may so happen that for two different pairs, these lay-outs may coincide for some region, and in that case, the one needs to increase the *height* of the chip since one of the wires has to be vertically above the other. One doesn't one the chip to be too high, and so it is a very important problem of how to choose the wiring for all necessary pairs, so as to keep the height to the minimum.

  To abstract the above example out, imagine we have a graph $G = (V, E)$ where the vertices correspond to various junctions in the chip including the pins, and the edges correspond to junctions between which a wire can be connected/placed. We also have k different *pairs* $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$,

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These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

namely the pins that need to be connected. For each such pair (s_i, t_i), we let $\mathcal{P}_i$ denote the collection of paths between s_i and t_i which can be used to connect; we assume the set $\mathcal{P}_i$ is explicitly given to us, and for this lecture we assume it is not too large. The objective is to choose a path $q^{(i)}$ from each $\mathcal{P}_i$ to lead to a solution $\mathcal{S} := \{q^{(1)}, \ldots, q^{(k)}\}$. Given this solution, the congestion $\mathsf{cong}(e)$ of an edge $e \in E$ is the number of paths $q^{(i)}$ in which e occurs. The congestion of the solution $\mathsf{cong}(S) := \max_{e\in E} \mathsf{cong}(e)$. The objective is to find the solution with the smallest congestion.

- *LP Relaxation.* The LP relaxation is hopefully clear.

$$\mathsf{lp} := \text{minimize} \quad \lambda \tag{CongMin-LP}$$

$$\sum_{p\in\mathcal{P}_i} x_{i,p} = 1, \qquad \forall 1 \le i \le k \tag{1}$$

$$\underbrace{\sum_{i=1}^{k} \sum_{p\in\mathcal{P}_i : e\in p} x_{i,p}}_{=\mathsf{cong}_x(e)} \le \lambda, \qquad \forall e \in E \tag{2}$$

$$x_{i,p} \ge 0, \qquad \forall 1 \le i \le k, \forall p \in \mathcal{P}_i \tag{3}$$

For each $i \in [k]$ and each $p \in \mathcal{P}_i$, we have a variable $x_{i,p}$ indicating whether $p \in \mathcal{P}_i$ is picked. (1) asserts exactly one path is chosen. (2) asserts that the congestion on every edge due to the solution x, denoted as $\mathsf{cong}_x(e)$, is at most λ, and the objective wishes to minimize this quantity. Since the objective is trying to minimize this value, for any $x_{i,p}$'s, λ would indeed be set to $\max_e \mathsf{cong}_x(e)$. In particular, if $x_{i,p} \in \{0, 1\}$, this would exactly capture the problem.

- *Randomized Rounding and Analysis via Chernoff Bounds.* Indeed, the rounding algorithm should also be clear : for each i, consider $x_{i,p}$ as a probability distribution over $\mathcal{P}_i$, and pick a path according to this distribution.

1: **procedure** CONG MINIMIZATION($G = (V, E)$, $\{s_i, t_i, \mathcal{P}_i\}_{i=1\ldots k}$):
2: Solve (CongMin-LP) to get $x_{i,p}$.
3: For each $i = 1, \ldots, k$, *independently* select exactly $q^{(i)} \in \mathcal{P}_i$ with probability $x_{i,q^{(i)}}$.

We analyze the algorithm in two regimes (applying two different regimes of Theorem 1).

Theorem 2. Let alg be the random variable indicating the maximum congestion of CONG MINIMIZATION. Then

- If $\mathsf{lp} \ge \ln m$, then $\mathbf{Pr}[\mathsf{alg} \ge \mathsf{lp} + 4\sqrt{\mathsf{lp}\ln m}] \le \frac{1}{m^3}$.
- If $\mathsf{lp} \le \ln m$, then $\mathbf{Pr}[\mathsf{alg} \ge \mathsf{lp} + \left(\frac{8\ln m}{\ln\ln m}\right) \cdot \mathsf{lp}] \le \frac{1}{m^3}$.

Here m is the number of edges in G.

Remark: *So if* $\mathsf{lp} \gg \ln m$, *then the congestion is* $(1+o(1))\mathsf{lp}$, *and in any case, it is a* $O(\ln m / \ln\ln m)$-*approximation.*

Proof. Fix an edge $e \in E$. We upper bound the probability that the congestion on this edge is too large. Then we apply the union bound.

For a fixed i, let $\mathcal{P}_i(e) := \{p \in \mathcal{P}_i : e \in p\}$. Define a Bernoulli random variable $X_i(e)$ which is 1 if the path $q^{(i)} \in \mathcal{P}_i(e)$ and 0 otherwise. Note that $\mathbf{Pr}[X_i(e) = 1] = \sum_{p \in P_i(e)} x_{i,p}$, which is $\leq \mathsf{lp}$ due to Equation (2). Crucially observe that $X_i(e)$'s, for $1 \leq i \leq k$, are mutually independent. Furthermore, $X(e) := \sum_{i=1}^k X_i(e)$ is the congestion on edge e induced by the algorithm. And thus,

$$\mathbf{Exp}[X(e)] = \sum_{i=1}^{k} \mathbf{Exp}[X_i(e)] = \sum_{i=1}^{k} \sum_{p \in P_i(e)} x_{i,p} \underbrace{\leq}_{\text{Equation (2)}} \mathsf{lp}$$

Now depending on how big lp is, we can apply the relevant Chernoff bound.

If $\mathsf{lp} \geq \ln m$, then we can apply (UT2) to get

$$\mathbf{Pr}\left[X(e) \geq \left(1 + \frac{4\sqrt{\ln m}}{\sqrt{\mathsf{lp}}}\right) \cdot \mathsf{lp}\right] \leq e^{-4\ln m} = \frac{1}{m^4}$$

If $\mathsf{lp} \leq \ln m$, we apply (UT3) with $t = \frac{8\ln m}{\ln\ln m}$

$$\mathbf{Pr}\left[X(e) \geq \mathsf{lp} + \left(\frac{8\ln m}{\ln\ln m}\right) \cdot \mathsf{lp}\right] \leq e^{-4\ln m \mathsf{lp}} \leq \frac{1}{m^4}$$

Applying the union bound over all m edges, we get the theorem. □

Exercise: ☕☕ *Unlike other randomized approximation algorithms, the above theorem is not a statement about* $\mathbf{Exp}[\mathsf{alg}]$. *Prove a bound on* $\mathbf{Exp}[\mathsf{alg}]$. *You may find the following identity useful:* $\mathbf{Exp}[Z] = \int_0^\infty \mathbf{Pr}[Z \geq t]dt$ *for any non-negative random variable.*

Notes

References